Example

Find the 4th Maclaurin polynomial (the Taylor polynomial of order 3 at point $x = 0$) generated by the function $f(x) = \frac{1}{1 + 2x}$. Also find the general nth term of Maclaurin series (excluding zero terms and assuming the Maclaurin series begins at $n = 0$).

The Taylor polynomial $P_3$ of order 3 of a function $f(x)$ about the point $x = 0$ is given by

$$P_3(x) = f(0) + f'(0)(x - 0) + f''(0)\frac{(x - 0)^2}{2!} + f'''(0)\frac{(x - 0)^3}{3!}.$$

First compute the appropriate derivatives. If $f(x) = \frac{1}{1 + 2x}$, then

$$f'(x) = -\frac{2}{(1 + 2x)^2}$$
$$f''(x) = \frac{8}{(1 + 2x)^3}$$
$$f'''(x) = -\frac{48}{(1 + 2x)^4}$$

using chain rule. Evaluating at $x = 0$ for use in the formula above,

$$f(0) = 1 \quad f'(0) = -2 \quad f''(0) = 8 \quad f'''(0) = -48.$$

Thus,

$$P_3(x) = 1 - 2x + 8x^2 \frac{x^2}{2!} - 48x^3 \frac{x^3}{3!} = 1 - 2x + 4x^2 - 8x^3.$$

Notice that $f^{(n)}(0) = n! \cdot (-2)^n$, for $n = 0, 1, 2, \ldots$

It follows that the (non-zero) nth term of the Maclaurin series $\sum_{n=0}^{\infty} f^{(n)}(0) \cdot \frac{x^n}{n!}$ is

$$(-2x)^n.$$